The generalized CBF

Generalized pairs:

A generalized sub-pair is the datum of $(X, B_X+M_X)/S$ s.t.:

- $X \longrightarrow S$ proper, X normal
- $\pi: X' \longrightarrow X$ birat'l proper
- $M_{X'}$ $\mathbb{Q}$-Cartier and nef$/S$ s.t. $M_X = \pi_* M_{X'}$
- $K_X + B_X + M_X$ $\mathbb{Q}$-Cartier

If $B_X \geq 0$, we call it generalized pair.

<u>Note:</u> we can replace X' and $M_{X'}$ with any X'' and $M_{X''}$ s.t.

$$M_{X''} := \varphi^* M_{X'} \qquad \begin{tikzcd} X'' \arrow[d] \arrow[dr, "\varphi"] & \\ X & X' \arrow[l, "\pi"] \end{tikzcd}$$

(the datum of $M_{X'}$ is the datum of a b-$\mathbb{Q}$-Cartier b-divisor b-nef$/S$)

Given any $\sigma: \widetilde{X} \longrightarrow X$ birational, we may assume $X' \longrightarrow \widetilde{X} \longrightarrow X$, thus we can define

$M_{\widetilde{X}} = \sigma_* M_{X'}$

$B_{\widetilde{X}}$ via: $K_{\widetilde{X}} + B_{\widetilde{X}} + M_{\widetilde{X}} = \sigma^*(K_X + B_X + M_X)$

We say (X, B_X+M_X) is g-(sub)-klt (g(sub)-lc) if $B_{\widetilde{X}} = B_{\widetilde{X}}^{<1}$ ($B_{\widetilde{X}} = B_{\widetilde{X}}^{\leq 1}$) for all such $\widetilde{X}$.

Note: (X, B_X+M_X) is g-klt (g-lc) $\Longleftrightarrow$ $(X', B_{X'})$ is sub-klt (sub-lc)

Note: assume M_X is $\mathbb{Q}$-Cartier then $M_{X'} \leq \pi^* M_X$ by neg-lemma

($M_{X'} = \pi^* M_X - E$)

Example: $(X, B_X+M_X) = (\mathbb{P}^2, B=0 \ \ M = 10L)$

$X' = Bl_p \mathbb{P}^2$, $M' = \underbrace{\pi^* M - 2E}_{\text{ample}}$

$\pi^*(K_X + B_X) = \pi^*(K_{\mathbb{P}^2}) = K_{X'} - E$

$\pi^* M = M' + 2E$

$\pi^*(K_X + B + M) = K_{X'} + E + M'$

strictly g-lc

Recall: Given a fibration which falls under the assumptions of the CBF, then

$$\begin{array}{c}(Y,\Delta)\\ \downarrow f\\ X\end{array}$$

X is endowed with the structure of a g-pair (X, B_X+M_X).

Today: if (X, B_X+M_X) has nice singularities, and $f: (X, B_X+M_X) \to Z$ with $K_X+B_X+M_X \sim_{\mathbb{Q}} 0/Z$, can we still have a CBF?

Why care?

- pairs (X, B_X) with $-(K_X+B_X)$ nef have interesting geometry, and setting $X=X'$, $M_X=-(K_X+B_X)$ we can turn them into a gpair of log CY type.
- approach Shokurov's conjectures inductively

$$\begin{tikzcd} (Y,\Delta) \arrow[d, "f"'] \arrow[dr] & \\ (W, B_W+M_W) & (X, B_X+M_X) \arrow[l] \end{tikzcd}$$

- g-pairs are interesting objects themselves

Setup: $(X, B_X + M_X)/S$ g-sub-pair

$f: X \longrightarrow Z$ s.t. $f_* \mathcal{O}_X = \mathcal{O}_Z$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Z \arrow[dl] \\ & S \end{tikzcd}$$

s.t. $K_X + B_X + M_X \sim_{\mathbb{Q}} f^* L_Z$

$\mathbb{Q}$-Cartier div L_Z on Z

For each $P \subseteq Z$ prime, we consider

$$g\text{-}lct_{\eta_P}(X, B_X + M_X; f^* P) = lct_{\eta_P}(X', B_{X'}; (f')^* P)$$

$$B_Z := \sum_{\substack{P \subset Z \\ \text{prime}}} \left(1 - glct_{\eta_P}(X, B_X + M_X; f^* P)\right) P$$

$$M_Z := L_Z - (K_Z + B_Z)$$

$$\begin{tikzcd} (X, B_X + M_X) \arrow[d, "f"'] & (\tilde{X}, B_{\tilde{X}} + M_{\tilde{X}}) \arrow[l, "\alpha"'] \arrow[d, "\tilde{f}"] \\ Z & Z' \arrow[l, "\beta"'] \end{tikzcd}$$

$K_{\tilde{X}} + B_{\tilde{X}} + M_{\tilde{X}} = \alpha^*(K_X + B_X + M_X)$

$L_{Z'} = \beta^* L_Z$

REDO THE ABOVE with $L_{Z'}$, $\tilde{f}$ and $(\tilde{X}, B_{\tilde{X}} + M_{\tilde{X}})$,

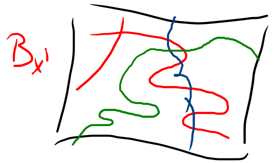
we get $B_{\tilde{Z}}$, $M_{\tilde{Z}}$ s.t. $\beta_* B_{\tilde{Z}} = B_Z$, $\beta_* M_{\tilde{Z}} = M_Z$

GOAL: show this process defines a g-pair $(Z, B_Z + M_Z)/S$

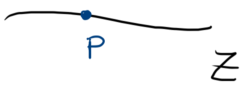
SETUP: X will be projective$/S$, $(X, B_X + M_X)/S$ g-lc
$$\begin{array}{c} (X, B_X+M_X)/S \\ \downarrow \\ Z \end{array}$$

(note: over η_Z $B_X^h \geq 0$)

IDEA: assume that $M_{X'}$ is semiample



$$\begin{array}{ccc} (X', B_{X'} + M_{X'}) & & \\ & \searrow^{\pi} & \\ & & (X, B_X + M_X) \\ \downarrow f' & \swarrow f & \\ Z & & \end{array}$$



pick $0 \leq D' \sim_{\mathbb{Q}} M_{X'}$

s.t. $\mathrm{lct}_{\eta_P}(X', B_{X'}; f^*P)$
$$\|$$
$\mathrm{lct}_{\eta_P}(X', B_{X'} + D'; f^*P)$

FACTS:

- CBF applies to $(X', B_{X'} + D') \longrightarrow Z$
- $B_Z \leq B_Z^{D'}$, $M_Z \geq M_Z^{D'}$
- $B_{\tilde{Z}} = \inf\limits_{D' \in |M_{X'}|_{\mathbb{Q}}} B_{\tilde{Z}}^{D'}$, $M_{\tilde{Z}} = \sup\limits_{D' \in M_{X'}|_{\mathbb{Q}}} M_{\tilde{Z}}^{D'}$

CONSEQUENCES: • if $\tilde{Z} \xrightarrow{\beta} \hat{Z}$, then

$$K_{\tilde{Z}} + B_{\tilde{Z}} \geq \beta^*(K_{\hat{Z}} + B_{\hat{Z}})$$

$$M_{\tilde{Z}} \leq \beta^* M_{\hat{Z}}$$

• finite base change property holds

MISSING: (☆) there is $\hat{Z}$ s.t., if $\tilde{Z} \xrightarrow{\beta} \hat{Z}$, then $\beta^*(K_{\hat{Z}} + B_{\hat{Z}}) = K_{\tilde{Z}} + B_{\tilde{Z}}$

(☆☆) $M_{\hat{Z}}$ nef $/S$

SKETCH: by the base change property, we may check these facts after a base change

By weak semistable reduction, we may assume that

$$\begin{array}{c} (X', \mathrm{Supp}((B')^h)) \\ \downarrow \\ Z \end{array}$$

is a family of slc pairs, Z smooth.

Once we have this reduction, it is a direct computation to show that $K_Z + B_Z$ stabilizes (the fiber product of a "nice family" is still nice, and $K_{X/Z}$ behaves well under fiber product)

This settles (☆)

Once (☆) is done, one can reduce (☆☆) to $\dim Z = 2$ computation.

What if $M_{X'}$ is not semiample?

Since $M_{X'}$ is nef$/Z$, if it is semiample$/Z$, we can get semiampleness by twisting by an ample on Z. A limiting argument would do.

In general $\nearrow$ $M_X|_{X_\eta} \equiv 0$ (1)

$\searrow$ $M_X|_{X_\eta} \not\equiv 0$ and is pseff (2)

If (1) holds, we may find

$$\begin{array}{ccc}
(X, B_X + M_X) & \longleftarrow & (X', B_{X'} + M_{X'}) \\
f \downarrow & & \downarrow f' \\
Z & \longleftarrow & Z'
\end{array}
\qquad M_{X'} \sim_{\mathbb{Q}} (f')^* \text{nef on } Z'$$

Use standard CBF for $(X', B_{X'}) \longrightarrow Z'$

If (2), we replace (X, B_X+M_X) with a gdlt model (we just need to make sure MMP for $K_X+B_X^h$ can be run)

Since $K_{X_\eta}+B_{X_\eta}+M_{X_\eta} \sim_{\mathbb{Q}} 0$, M_{X_η} pseff not 0,

$-(K_{X_\eta}+B_{X_\eta})$ pseff not 0

$\Rightarrow K_X+B_X^h$ not pseff$/Z$

Run $(K_X+B_X^h)$-MMP$/Z$

$$
\begin{tikzcd}
 & & X' \arrow[dll] \arrow[d] \\
(X, B_X+M_X) \arrow[rr, dashed] \arrow[ddr, "f"'] & & (\tilde{X}, B_{\tilde{X}}+M_{\tilde{X}}) \arrow[d, "g"] \\
 & & (W, B_W+M_W) \arrow[dl] \\
 & Z &
\end{tikzcd}
$$

MFS $\quad \rho(\tilde{X}/W)=1$

$\Rightarrow M_{\tilde{X}}$ ample$/W$

By perturbation argument, we reduce to the case $M_{X'}$ is semiample$/W$

$\Rightarrow$ apply g-CBF to $\tilde{X} \longrightarrow W$

If $W \longrightarrow Z$ not birat'l, do induction on rel dim